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Long-time behavior of continuous time models in genetic algebras

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Abstract. In [2] the solutions of Andreoli's differential equation in genetic algebras with genetic realization were shown to converge to equilibria. Here we derive an explicit formula for these limits.

Key words: Continuously overlapping generations – Genetic algebra – Long-time behavior.

1 Continuous time models and genetic algebras

Algebras in genetics have attracted considerable interest in recent years. Both discrete and continuous processes are studied. Here we will deal with the latter ones. In [1], Andreoli introduced the differential equation

$$\begin{aligned} \dot{x} &= x^2 - x\,, \\ x(0) &= y \end{aligned}$$

in a genetic algebra to model the time dependence of the genotype frequencies of a population in the limiting case of continuously overlapping generations. Heuch [3] showed that this equation can be solved by elementary functions. A first result on the long-time behavior was proved in Wörz-Busekros [6]. In [2] it was shown that for genetic algebras with genetic realization the solution converges to an idempotent of the algebra (a property that also holds for Bernstein algebras for which a closed formula for the solution was given). The question of how to compute the limit efficiently had to remain open there. This paper is an attempt to fill this gap. The main result is an explicit formula for the limit of the solution of the differential equation in question.

We first summarize some facts which were proved in [2] and [5]:

Proposition 1.1 *Let $G(y, t)$ resp. $S(y, t)$ be the solution of $\dot{x} = x^2$ resp. $\dot{x} = x^2 - x$ with $x(0) = y$ in a real or complex commutative algebra A of finite dimension.*

a) $S(y,t) = e^{-t} \cdot G(y, 1-e^{-t})$, *wherever both sides are defined.*
b) *Let* $G(y,t) = \sum_{k \geqq 0} t^k g_k(y)$ *be the Taylor expansion about* $t = 0$. *Then*

$$(k+1) g_{k+1}(y) = D g_k(y) y^2 = \sum_{j=0}^{k} g_j(y) g_{k-j}(y)$$

for all $k \geqq 0$. *In particular, for all* $k \geqq 0, g_k$ *is a homogeneous polynomial map of degree* $k+1 : g_0(y) = y,\ g_1(y) = y^2,\ g_2(y) = y^3,\ g_3(y) = \frac{1}{3}(2y^4 + y^2 y^2)$ *etc.*
c) $\lim_{t \to \infty} S(y,t) = \lim_{k \to \infty} g_k(y)$, *whenever the right hand side exists.*
d) *For any homomorphism* $\omega : A \to \mathbb{K}$, $\omega(g_k(y)) = \omega(y)^{k+1}$ *for all* $y \in A$ *and* $k \geqq 0$.

Now let $\mathbb{K}$ denote the field of real or complex numbers and A a finite dimensional commutative $\mathbb{K}$-algebra with left multiplication $L(x)$. Then A is called a *genetic algebra* if there is a nontrivial homomorphism $\omega : A \to \mathbb{K}$ and the coefficients of the characteristic polynomial of any transformation $f(L(x_1), \ldots, L(x_s))$ only depend on $\omega(x_1), \ldots, \omega(x_s)$ for any polynomial f in s noncommuting indeterminates; the second requirement is equivalent to A (in the real case $A \otimes \mathbb{C}$) permitting coordinates such that $L(x)$ is lower triangular for all $x \in A$ and strictly lower triangular for all $x \in \operatorname{Ker} \omega$ (see [6] for details). Hence there is (at least in $A \otimes \mathbb{C}$) a basis $(c_0, \ldots, c_n)$ such that

$$c_0^2 = c_0,$$

$$c_0 c_i = \lambda_i c_i + \sum_{k=i+1}^{n} \lambda_{0ik} c_k \text{ for } 1 \leqq i \leqq n,$$

$$c_i c_j = \sum_{l=\max\{i,j\}+1}^{n} \lambda_{ijl} c_l \text{ for } 1 \leqq i, j \leqq n$$

with $\lambda_i,\ \lambda_{0ik},\ \lambda_{ijl} \in \mathbb{C}$ (cf. [6, Theorem 3.13]). The numbers $\lambda_0 = 1, \lambda_1, \ldots, \lambda_n$ are usually called the train roots of A. If $S(y,t)$ is bounded for $t \to \infty$, then $S(y,t)$ converges to an idempotent of A; cf. [2]. This holds in particular if A has a *genetic realization*, i.e. if there are coordinates $x_0, \ldots, x_n$ such that $\omega(x) = x_0 + \cdots + x_n$ and the standard simplex $S = \{x \in A : x_i \geqq 0,\ \omega(x) = 1\}$ is closed under multiplication in A.

2 A train equation for genetic algebras

It is well-known that genetic algebras are train algebras, i.e. there is a positive integer s and $\alpha_1, \ldots, \alpha_s \in \mathbb{K}$ such that the baric identity

$$x^{s+1} + \alpha_1 \omega(x) x^s + \cdots + \alpha_s \omega(x)^s x = 0$$

holds for all $x \in A$; cf. [6].

We will show that the $g_k(x)$ introduced above also satisfy a baric identity:

Proposition 2.1 *Let* A *be a genetic algebra. Then there is a positive integer* r *and* $\gamma_1, \ldots, \gamma_r \in \mathbb{K}$ *such that*

$$g_r(x) + \gamma_1 \omega(x) g_{r-1}(x) + \cdots + \gamma_r \omega(x)^r g_0(x) = 0$$

for all $x \in A$.

Proof. As cited above, we can find coordinates $(x_0, x_1, \ldots, x_n), n+1 = \dim A$, such that

$$g_1(x) = x^2 = \begin{pmatrix} & & x_0^2 \\ p_1(x_0) & + & \lambda_1 \cdot x_0 x_1 \\ p_2(x_0, x_1) & + & \lambda_2 \cdot x_0 x_2 \\ & \vdots & \\ p_n(x_0, x_1, \ldots, x_{n-1}) & + & \lambda_n \cdot x_0 x_n \end{pmatrix}$$

where $p_1, \ldots, p_n$ are homogeneous quadratic polynomials. We introduce a new grading on the polynomial ring $\mathbb{K}[x_0, x_1, \ldots, x_n]$ by assigning

$$\deg x_i = \begin{cases} 0, & \text{if } i = 0 \\ 2^{i-1}, & \text{if } 1 \leqq i \leqq n. \end{cases}$$

With this definition, one checks that

$$\deg p_j(x_0, \ldots, x_{j-1}) \leqq 2^j$$

for all j, and thus

$$\deg g_1(x) \leqq \begin{pmatrix} 0 \\ 2^0 \\ 2^1 \\ \vdots \\ 2^{n-1} \end{pmatrix},$$

where "$\leqq$" is to be understood componentwise. We claim that this holds in general, i.e.

$$\deg g_k(x) \leqq \begin{pmatrix} 0 \\ 2^0 \\ 2^1 \\ \vdots \\ 2^{n-1} \end{pmatrix},$$

for all $k \in \mathbb{N}$.
Assume, it holds for some $k \geqq 1$. Then

$$\deg D_{g_k}(x) \leqq \begin{pmatrix} 0 & 0 & 0 & & & \\ 2^0 & 0 & 0 & & 0 & \\ 2^1 & 2^1 - 2^0 & 0 & & & \\ 2^2 & 2^2 - 2^0 & 2^2 - 2^1 & \ddots & & \\ \vdots & \vdots & \vdots & \ddots & 0 & \\ 2^{n-1} & 2^{n-1} - 2^0 & 2^{n-1} - 2^1 & \ldots & 2^{n-1} - 2^{n-2} & 0 \end{pmatrix}.$$

Since

$$g_{k+1}(x) = \frac{1}{k+1} D_{gk}(x) \cdot x^2 \qquad \text{(cf. Prop. 1.1)},$$

$$\deg\, g_{k+1}(x) \leqq \begin{pmatrix} 0 \\ 2^0 \\ 2^1 \\ \vdots \\ 2^{n-1} \end{pmatrix},$$

as desired.

But the set of monomials of degree not greater than 2^{n-1} is – up to powers of x_0 – finite. Thus, the g_k contain – up to powers of x_0 – only a finite number M of monomials. For $N = M \cdot \dim A$ the polynomials $\omega^N g_0, \omega^{N-1} g_1, \ldots, g_N$ will therefore be linearly dependent over $\mathbb{K}$. Now choose r minimal such that $\omega^r g_0, \omega^{r-1} g_1, \ldots, g_r$ are lineraly dependent over $\mathbb{K}$. □

The converse is in general false, i.e. an algebra satisfying a train equation for the g_k is not necessarily genetic. A counterexample is the algebra constructed in the proof of Theorem 3.21 in [6] to show that an algebra satisfying a "usual" train equation is not genetic in general. Let N be Suttle's famous 5-dimensional commutative algebra that is nil, but not nilpotent; cf. [4]. In particular, $x^4 = x^2 x^2 = 0$ for all $x \in N$. Define the algebra $A := \mathbb{K} \cdot c \oplus N$ by adjoining an idempotent c to N via $(x_0 \cdot c + x)^2 := x_0^2 \cdot c + x_0 \cdot x + x^2$ for all $x_0 \in \mathbb{K}$ and $x \in N$. Note that $\omega(x_0 \cdot c + x) := x_0$ defines a nontrivial homomorphism of A. It is straightforward to verify that $g_3(y) - 2\omega(y) g_2(y) + \frac{7}{6}\omega(y)^2 g_1(y) - \frac{1}{6}\omega(y)^3 g_0(y) = 0$ holds for all $y \in A$.

Proposition 2.1 will now be used to compute the desired limit. We start with an auxiliary result:

Proposition 2.2 *Let A be a commutative $\mathbb{K}$-algebra satisfying a train equation $g_r(x) + \gamma_1 \omega(x) g_{r-1}(x) + \cdots + \gamma_r \omega(x)^r g_0(x) = 0$. Then for any positive integer k there are $\gamma_{k,l}, \ldots, \gamma_{k,r} \in \mathbb{K}$ such that*

$$g_{r+k}(x) + \gamma_{k,1} \omega(x)^{k+1} g_{r-1}(x) + \cdots + \gamma_{k,r} \omega(x)^{k+r} g_0(x) = 0$$

for all $x \in A$.

Proof. Induction on k does the trick, the case $k = 0$ being the hypothesis. Now let $k > 0$ and differentiate with respect to x in direction of x^2, using $(k+1) g_{k+1}(x) = Dg_k(x) x^2$:

$$\begin{aligned}
&(r+k+1) g_{r+k+1}(x) + r\gamma_{k,1} \omega(x)^{k+1} g_r(x) + \\
&\quad + ((k+1)\gamma_{k,1} + (r-1)\gamma_{k,2}) \omega(x)^{k+2} g_{r-1}(x) + \\
&\quad + ((k+2)\gamma_{k,2} + (r-2)\gamma_{k,3}) \omega(x)^{k+3} g_{r-2}(x) + \cdots + \\
&\quad + ((k+r-1)\gamma_{k,r-1} + \gamma_{k,r}) \omega(x)^{k+r} g_1(x) + \\
&\quad + (k+r)\gamma_{k,r} \omega(x)^{k+r+1} g_0(x) = 0.
\end{aligned}$$

Applying the hypothesis on $g_r(x)$ yields the result. □

3 The limit formula

Let $H := \{x \in A : \omega(x) = 1\}$. What Proposition 2.2 tells us is that if the limit $g(x) := \lim_{k \to \infty} g_k(x)$ exists for $x \in H$, it can be represented as a linear combination of $g_0(x),\ g_1(x), \ldots, g_{r-1}(x)$, say

$$g(x) + \alpha_1 g_{r-1}(x) + \cdots + \alpha_{r-1} g_1(x) + \alpha_r g_0(x) = 0$$

for all $x \in H$. Since $g(x)$ is constant along the trajectory $S(x, t)$, it is an invariant of motion, i.e.

$$Dg(x) \cdot (x^2 - x) = 0$$

for all $x \in H$. Combining the two equations yields

$$\sum_{i=1}^{r} \alpha_i Dg_{r-i}(x) x^2 = \sum_{i=1}^{r} \alpha_i Dg_{r-i}(x) x.$$

Applying Proposition 2.1 allows us to simplify to

$$\sum_{i=1}^{r} \alpha_i (r + 1 - i) g_{r+1-i}(x) = \sum_{i=1}^{r} \alpha_i (r - i) g_{r-i}(x).$$

For arbitrary y with $w(y) \neq 0$, $\frac{y}{\omega(y)} \in H$, thus:

$$\sum_{i=1}^{r} \alpha_i (r + 1 - i) \omega(y)^{i-1} g_{r+1-i}(y) = \sum_{i=1}^{r} \alpha_i (r - i) \omega(y)^{i} g_{r-i}(y).$$

Since the set $\{y \in A : \omega(y) \neq 0\}$ is Zariski-open, the last equation is an identity in A. Now we combine corresponding terms and apply the train equation for the g_k:

$$\begin{gathered}
(-r\alpha_1 - r\gamma_1\alpha_1 + (r-1)\alpha_2)\omega(y) g_{r-1}(y) + \\
(-r\gamma_2\alpha_1 - (r-1)\alpha_2 + (r-2)\alpha_3)\omega(y)^2 g_{r-2}(y) + \\
\vdots \\
(-r\gamma_{r-1}\alpha_1 - 2\alpha_{r-1} - \alpha_r)\omega(y)^{r-1} g_1(y) + \\
(-r\gamma_r\alpha_1 - \alpha_r)\omega(y)^r g_0(y) = 0.
\end{gathered}$$

Since the polynomials $\omega^r \cdot g_0, \ldots, \omega \cdot g_{r-1}$ are linearly independent over $\mathbb{K}$, we can equate the coefficients to zero, thus deriving the following system of linear equations:

$$\begin{pmatrix}
-r - r\gamma_1 & r-1 & 0 & 0 & \ldots & 0 \\
-r\gamma_2 & -r+1 & r-2 & 0 & \ldots & 0 \\
-r\gamma_3 & 0 & -r+2 & r-3 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ldots & -3 & 2 & 0 \\
-r\gamma_{r-1} & 0 & \ldots & 0 & -2 & 1 \\
-r\gamma_r & 0 & \ldots & \ldots & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \vdots \\ \vdots \\ \alpha_r
\end{pmatrix} = 0.$$

The solution can be computed easily:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \alpha \cdot \begin{pmatrix} \frac{1}{r}(\gamma_1 + \gamma_2 + \ldots + \gamma_r) \\ \frac{1}{r-1}(\gamma_2 + \ldots + \gamma_r) \\ \vdots \\ \frac{1}{2}(\gamma_{r-1} + \gamma_r) \\ \gamma_r \end{pmatrix},$$

for some constant $\alpha \in \mathbb{K}$. Finally, the constant α can be determined from the fact that $\omega(g_k(x)) = 1$ for all $x \in H$ and $k \geqq 0$ (see Proposition 1.1) and, as a consequence, $\omega(\lim_{k\to\infty} g_k(x)) = 1$. Working this out finally yields:

Theorem 3.1 *Let A be a commutative $\mathbb{K}$-algebra with a nontrivial homomorphism $\omega : A \to \mathbb{K}$ satisfying the identity*

$$g_r(x) + \gamma_1\omega(x)g_{r-1}(x) + \cdots + \gamma_r\omega(x)^r g_0(x) = 0$$

for some $\gamma_1, \ldots, \gamma_r \in \mathbb{K}$. If for $x \in A$ with $\omega(x) = 1$ the limit $\lim_{k\to\infty} g_k(x)$ exists, then

$$\lim_{k\to\infty} g_k(x) = \frac{\sum_{i=1}^{r}\left(\frac{1}{r+1-i}\sum_{j=i}^{r}\gamma_j\right) g_{r-i}(x)}{\sum_{i=1}^{r}\left(\frac{1}{r+1-i}\sum_{j=i}^{r}\gamma_j\right)}$$

which coincides with $\lim_{t\to\infty} S(x, t)$.

It should be noted that it is not clear if existence of the limit of $S(x, t)$ for $t \to \infty$ implies the existence of the limit of $g_k(x)$ for $k \to \infty$.

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